A general spatiotemporal equilibrium model of ride-hail market

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1 Introduction

The rapid growth of transportation network companies (TNCs) have reshaped the ride-hail industry in the past decade. Taking advantage of mobile technologies, the e-hail service offered by TNCs has quickly gained popularity, rivaling conventional taxis around the world. E-hail is celebrated for improving passengers' travel experiences in the low-density areas that have been poorly covered by traditional taxi services. Yet, properly serving these areas requires recognizing the system as an interlinked network of local markets with varying supply-demand conditions. Clearly, drivers prefer to cruise in a local market with a greater probability of meeting a passenger and a higher average trip fare. While the latter is determined by demand and pricing, the former is sensitive to the collective search behaviors of idle drivers. The distribution of idle drivers in the network, in turn, affects the system performance (e.g., the passenger wait time in each local market). Therefore, the platform needs to develop its operational strategies, such as pricing, by anticipating drivers' movements across local markets. In this paper, we develop a spatiotemporal equilibrium model of a ride-hail market defined on a network of local markets. The model considers a e-hail platform or a taxi operator monopolize the market, and takes its pricing and matching strategies as given. The goal is to predict the distribution and movements of idle drivers in the network, assuming each driver makes his relocation decision to maximize his own expected return. Although not considered in this paper, the proposed model lays a foundation for evaluating and designing the platform's operational strategies that may vary over space and time (e.g., surge pricing).

Extensive research efforts have been devoted to modeling spatial equilibrium in a ride-hail market. Studies in the literature may be broadly classified as decentralized (e.g., Yang and Wong, 1998; Lagos, 2000) and centralized models (e.g., Zhang and Pavone, 2016; Braverman et al., 2019), according to whether or not drivers are allowed to relocate at their own discretion. In the decentralized models, a driver is assumed to choose his next search location based on the expected

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return of each local market. The search strategy may be treated as deterministic (e.g., Lagos, 2000; Bimpikis et al., 2019) or stochastic (e.g., Yang and Wong, 1998; Yang et al., 2010). This study develops a decentralized model with a deterministic search strategy.

The passenger-driver matching process in a local market has also been modeled differently in existing studies. A common approach is to form a virtual queue of idle drivers, similar to a taxi stand (Zhang and Pavone, 2016; Banerjee et al., 2017; Braverman et al., 2019). Each arriving passenger is instantly picked up by the first idle driver, or leaves the system right away if the idle driver queue is empty. Hence, the detailed matching and pickup processes are not considered in these queuing network models. Another stream of studies ignores the matching friction, assuming the pickup number simply equals the minimum between the number passengers and the number of idle drivers (Lagos, 2000; Bimpikis et al., 2019). On the other hand, Yang et al. (2010) introduce an aggregate matching function to characterize the relationship between pickup rate and the number of waiting passengers and idle drivers. In this paper, the matching process is described by a new mechanism proposed in Chen et al. (2019) and empirically validated in Zhang et al. (2019). Specifically, this mechanism is applied to derived the likelihood of finding a passenger in a local market within a certain time period.

In the context of spatial equilibrium, most studies have focused on the steady state, leaving out the dynamic evolution of passenger demand and driver supply. Guda and Subramanian (2018) is a notable exception, which investigates the impact of surge pricing using a highly stylized model (i.e., two adjacent zones over two successive time periods). This paper addresses this gap by considering a more general network topology over multiple discrete time periods and derive the equilibrium distribution and flow pattern of idle drivers in the entire analysis horizon.

At the core of the proposed equilibrium model is a Markov decision process (MDP) used by each driver to optimize his search strategy. Unlike previous studies using a similar modeling framework (e.g., Yu et al., 2019; Shou et al., 2020), we assume every driver optimize his search strategy while anticipating the strategies of others. The model is thus a variant of MDP congestion game (Calderone and Sastry, 2017; Calderone and Shankar, 2017). Compared to Calderone and Sastry (2017) and Li et al. (2019), our model characterizes the "congestion" effect differently. In a local ride-hail market, congestion arises as idle drivers compete for passengers, thus it is captured by the probability of meeting a passenger, referred to as *meeting probability*. Among other things, this probability depends on the densities of idle drivers and waiting passengers as well as the underlying matching mechanism. As mentioned above, we will apply the mechanism developed in our previous studies to specify the meeting probability for both taxi and e-hail services.

To summarize the methodological contribution of this study,

- we develop a general spatiotemporal equilibrium model that captures the profit-driven search behaviors across a network of connected local ride-hail markets.

- we incorporate a physical matching mechanism in the equilibrium model and apply it to both taxi and e-hail services to characterize the network congestion effect in each local market created by the competition among idle drivers.

- we model the driver's search behaviors as an MDP and show that it leads to a congestion game analogous to the traffic assignment problem.

The next section presents the main methodology of this paper. Section 2.1 describes the dynamic spatial equilibrium as an MDP congestion game. Section 2.2 explains how the meeting probability is specified using a spatial matching theory. Section 2.3 outlines an solution algorithm for the equilibrium problem.

2 Methodology

2.1 Dynamic spatial equilibrium as an MDP routing game

Consider a ride-hail market divided into N zones (local markets) and an analysis horizon consisting of H discrete time periods of identical length Δ. The travel times between zone i and zone j, denoted as τ_{ij}, is assumed to be multiples of Δ and $\tau_{ii} = 0, \forall i$. Let $q_{i,t}$ be the demand rate originated from zone i during time period t, and α_{ij} be the fraction of demand from i destined for j ($\sum_j \alpha_{ij} = 1$). A fleet of M drivers travel across different zones of the market to search and deliver passengers. We use $y_{i,t}$ to denote the number of idle drivers in zone i at the beginning of time period t.

Matching is modeled as a process where idle drivers in each zone are assigned to passengers originating from the same zone. It starts at the beginning of each period, taking the passenger demand rate and the number idle drivers as inputs, and yields the matched and unmatched drivers at the end of period according to certain matching mechanism. The meeting probability during period t in zone i, denoted as $m_{i,t}$, is thus a function of $q_{i,t}$ and $y_{i,t}$ defined for a given type of ride-hail service, which will be specified in Section 2.2. At the end of period t, $m_{i,t}y_{i,t}$ drivers are successfully matched and ready to deliver passengers to their destinations. The others will decide their next search destination that maximizes their expected return. Therefore, the driver supply $y_{i,t}$ during period t includes those who were in zone i at the end of period $t-1$ (either drop off passengers or fail to meet passengers) and decide to search locally, and those who just arrive at zone i at t for passenger search. Let $x_{ij,t}$ denotes the relocation flow from i to j at t. Hence, $y_{i,t}$ can be written as

$$y_{i,t} = \sum_j x_{ji,t-\tau_{ji}}. \tag{1}$$

A driver's movements across zones over the analysis horizon is formulated as a Markov decision process (MDP) represented by a tuple $(\mathcal{S}, \mathcal{A}, T, R, \beta)$. The components are specified as follows:

- $\mathcal{S}$ is a set of states and each state $s = (i, t)$ is represented by the driver's location i at t before he makes the relocation decision.

- $\mathcal{A}$ is a set of actions and each action a is defined as the next search zone.

- T denotes a state transition probability function that is jointly determined by the meeting probability and the demand pattern:

$$T(s' = (k, t+\tau_{ij}) | s = (i,t), a = j) = \begin{cases} m_{j,t+\tau_{ij}}\alpha_{jk} & j \neq k \\ 1 - m_{j,t+\tau_{ij}} & j = k \end{cases}; \tag{2}$$

- R is a reward function of the current state, action and the following state, which is simply the discounted revenue if the driver is matched and zero otherwise:

$$R(s = (i,t), a = j, s' = (k, t + \tau_{ij})) = \begin{cases} \beta^{\tau_{ij}} p_{jk} & j \neq k \\ 0 & j = k \end{cases}; \tag{3}$$

- β is a discount factor $\beta \in (0, 1]$ that accounts for the preference for the present gain over the future gain.

Therefore, the maximum expected return for a driver in state s is characterized by the Bellman's optimality equation:

$$V(s) = \max_{a \in \mathcal{A}} \left[\sum_{s}^{\prime} T(s'|s,a)(R(s,a,s') + \beta^{\tau_{jk}+1} V(s')) \right]. \tag{4}$$

We note that Eq.(4) is derived based on the assumption that all drivers accept their assigned rides. It is straightforward to consider the case where drivers are allowed to reject requests, as recently implemented by Uber in California (Zipper, 2019). Also, the trip fair p_{ij} is differentiated by trip origin and destination, but kept constant over time. Again, it is easy to have it vary over time.

In this system, every driver makes decision based on the value function Eq.(4). Their collective behaviors lead to the relocation flow $\mathbf{x} = \{x_{ij,t}\}$ and the distribution of idle drivers $\mathbf{y} = \{y_{i,t}\}$, which, in turn, affects the meeting probability $m_{i,t}$ in Eq. (2) and consequently the value function. In other words, both the transition probability and the value function (Eq. (4)) are functions of $\mathbf{x}$. If all drivers have perfect information and consistently following the decision process described above, the system will eventually evolves to a state similar to the Wardrop equilibrium that has been widely applied in urban travel forecasting (Wardrop, 1952; Beckmann et al., 1956). At such a state, no individual driver could benefit from unilaterally changing his search strategy in any zone at any time. Finding this equilibrium belongs to a broad class of problems known as MDP congestion game (Calderone and Sastry, 2017; Calderone and Shankar, 2017).

Let $Q(s, a, \mathbf{x})$ be the expected return for a driver in state s taking action a given the strategies of others. Hence,

$$Q(s, a, \mathbf{x}) = \sum_{s} T(s'|s, a, \mathbf{x})(R(s, a, s') + \beta d(s, a, s') V(s', \mathbf{x})). \tag{5}$$

Mathematically, a flow pattern of idle drivers $\mathbf{x}^*$ is said to be a Wardrop equilibrium if for any zone i at time t such that $x^*_{ij,t} > 0$,

$$Q((i,t), j, \mathbf{x}^*) \geq Q((i,t), k, \mathbf{x}^*), \forall k. \tag{6}$$

2.2 Specification of meeting probability

Under mild assumptions, Chen et al. (2019) and Zhang et al. (2019) derive the waiting time distribution for street-hail and e-hail passengers. For passengers in zone i at time t, it is given by

$$\text{Taxi: } F^s_{i,t}(u) = 1 - \exp\left(-\frac{\sigma_{i,t} d_{i,t} v_{i,t} \Lambda_{i,t}}{\delta} u\right), \tag{7}$$

$$\text{E-hail: } F^e_{i,t}(u) = 1 - \exp\left(-\frac{\pi k_{i,t} v^2_{i,t} \Lambda_{i,t}}{\delta^2 \Pi_{i,t}} u^2\right), \tag{8}$$

where δ is the detour factor of road network, $v_{i,t}$ is the average cruising speed, $\Lambda_{i,t}$ is the vacant vehicle density (including idle and matched drivers), $\sigma_{i,t}$ reflects the attractiveness of certain location towards nearby idle drivers (or a concentration level of demand), $d_{i,t}$ denotes the maximum hail radius in street-hail, $\Pi_{i,t}$ refers to the waiting passenger density, and $k_{i,t}$ measures the efficiency of matching algorithm in e-hail.

By symmetry, we could obtain the distribution of driver's search time, which reads

$$\text{Taxi: } G^s_{i,t}(u) = 1 - \exp\left(-\frac{\sigma_{i,t} d_{i,t} v_{i,t} \Pi_{i,t}}{\delta} u\right), \tag{9}$$

$$\text{E-hail: } G^e_{i,t}(u) = 1 - \exp\left(-\frac{\pi k_{i,t} v^2_{i,t} \Pi_{i,t}}{\delta^2 \Lambda_{i,t}} u^2\right). \tag{10}$$

Therefore, the probability of meeting a passenger in zone i during period t is given by $m_{i,t} = G^z_{i,t}(\Delta),\ z \in \{s, e\}$. Suppose all zones have the same size $A = 1$, then the vacant vehicle density can be substituted by the number of idle drivers at the beginning of period t, i.e., $\Lambda_{i,t} = y_{i,t}/A = y_{i,t}$. As per Little's formula (Little, 1961), waiting passenger density may be approximated by $\Pi_{i,t} = q_{i,t}\bar{w}_{i,t}/A = q_{i,t}\bar{w}_{i,t}$, where $\bar{w}_{i,t}$ is the average passenger waiting time with close forms (Chen et al., 2019; Zhang et al., 2019):

$$\text{Taxi: } \bar{w}^s_{i,t} = \frac{\delta}{\sigma_{i,t} d_{i,t} v_{i,t} \Lambda_{i,t}}, \tag{11}$$

$$\text{E-hail: } \bar{w}^e_{i,t} = \frac{\delta}{2 v_{i,t}} \sqrt{\frac{\Pi_{i,t}}{k \Lambda_{i,t}}}. \tag{12}$$

With some algebra, we finally derive the meeting probabilities as follows:

$$\text{Taxi: } m^s_{i,t} = 1 - \exp\left(-\frac{q_{i,t}}{y_{i,t}} \Delta\right), \tag{13}$$

$$\text{E-hail: } m^e_{i,t} = 1 - \exp\left(-\frac{\pi}{4} \frac{q^2_{i,t}}{y^2_{i,t}} \Delta^2\right). \tag{14}$$

2.3 Solution algorithm

Conventional solution methods of MDP (e.g., backward induction) cannot be applied to solve the proposed model because the system dynamics characterized by the meeting probabilities change as drivers alter their search strategies. However, we notice that drivers' movements in the ride-hail market is analogous to those in a traffic network. Selfish travelers seek the fastest path to their destinations, while the accumulative flow creates congestions. Similarly, drivers relocate to areas that yield the maximum expected return, though the increase in supply drags down the meeting probability. Therefore, we apply the method of successive average (MSA) to solve the equilibrium, which has been widely applied to solve the stochastic user equilibrium (SUE) (e.g., Sheffi and Powell, 1982; Powell et al., 1995). In each iteration, we first update the value function using current relocation flows $\mathbf{x}$. Similar to all-or-nothing assignment, we solve the optimal policy and derive a moving direction of $\mathbf{x}$. We update $\mathbf{x}$ in the sense of successive average until the convergence criterion is achieved.

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